# An Algebraic Approach to Offsetting and Blending of Solids

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**Abstract**

We propose to broaden the framework of CSG to a representation of solids as Boolean combinations of polynomial equations and inequalities describing regular closed semialgebraic sets of points in 3-space. As intermediate results of our operations we admit arbitrary semialgebraic sets. This allows to overcome well-known problems with the computation of blendings via offsets. The operations commonly encountered in solid modelers plus offsetting and constant radius blending are reduced to quantifier elimination problems, which can be solved by exact symbolic methods. We discuss the general properties of such offsets and blendings for arbitrary regular closed semialgebraic sets in real $n$-space. In a collection of computational examples we demonstrate the capabilities of the REDLOG package for the discussed operations on solids within our framework.

*Key words:* Offsetting solids. Blending solids. Representation of solids.

## 1 Introduction

Solid modeling knows two major representation schemata for solid objects: *boundary representation* (B-rep) and *constructive solid geometry* (CSG). B-rep is very close to the image rendered on the screen. CSG, in contrast, resembles the inherent structure of the described solid by combining primitive solids via regularized Boolean operations. Today's modelers are usually dual representation modelers, which combine both schemata to make use of the particular advantages of each of them. [12].

We propose a generalization of CSG representing solids as certain Boolean combinations of polynomial equations and inequalities, where the solids are derived as the corresponding *semialgebraic sets* of points in 3-space satisfying these *formulas*. Among all semialgebraic sets, we define regular closed sets to be valid solids.

The treatment of various problems from computational geometry within our framework has already been discussed elsewhere [19]. This includes parallel and central projections, corresponding reconstruction problems, offsets of algebraic surfaces, Voronoi diagrams, and collision problems. From a theoretical point of view, it is clear that surface descriptions analogous to B-rep can be obtained from our representation. Current research is concerned with efficient strategies for doing so, and experimental implementations make this appear feasible.

This article focuses on offsets and blendings of solids. Our approach allows to naturally handle *arbitrary* semialgebraic sets as intermediate results of our operations. It shall turn out that this freedom, together with a new concept of defining blendings in terms of offsets, allows us to understand and to overcome some well-known problems with the definition of blendings via offsets [16,17].

Our notion of a solid obviously extends that of CSG. In particular, it includes common types of offsetted and blended solids. In fact, all operations commonly encountered in today's solid modelers do not lead outside the realm of semialgebraic sets. To show this, we have to redefine all desired operations to map formulas to formulas, and we have to give algorithms computing these maps.

It turns out that the operations can be straightforwardly performed if we allow the resulting formulas to involve *quantifiers* "$\exists x$" and "$\forall x$" as they are used, e.g., in definitions of elementary calculus. Maybe a little surprisingly, it is possible to map each formula $\varphi$ containing such quantifiers to an equivalent formula $\varphi'$ not containing any quantifier. Here, *equivalent* means that both $\varphi$ and $\varphi'$ describe the same semialgebraic set. For instance, $\varphi \equiv \exists x(ax + b = 0)$ is equivalent to $\varphi' \equiv a \neq 0 \vee b = 0$ with "$\vee$" denoting "or." Note that the variable $x$ quantified in $\varphi$ does not occur in $\varphi'$ at all. The step from $\varphi$ to $\varphi'$ is known as *quantifier elimination*, and there are various software packages performing quantifier elimination available [8]. Note that in contrast to purely algebraic approaches discussed elsewhere [12,13] quantifier elimination can handle polynomial inequalities both in the input and in the output.

It is a well-known experience in computational real algebra that real elimination methods are very limited in practice. We mainly use a method by the second author [15,24], which has stretched these limits considerably under the restriction that the polynomials to be handled are of low degree. It is implemented in the REDUCE [10] package REDLOG[1] by the first author and A. Dolzmann [3,5]. REDLOG has been successful in a variety of application areas where these restrictions are frequently satisfied, among them automatic theorem proving in geometry [7] and geometric projection problems [19].

Among the numerous variants of blending [11,21] we restrict ourselves here to circular constant radius blending that can be visualized as an envelope of a ball rolling along parts of the surfaces or solids. We discuss the general properties of such blendings reducing them to an iterated formation of certain offsets, where our approach differs from the usual concatenations of shrinking and expanding [16,17].

It is a crucial feature of our approach that it can solve problems parametrically. We can, e.g., compute the blend of a solid with respect to a parametric radius $r$. The resulting description will contain case distinctions that make it correct for any substitution of a positive real number for $r$.

In a collection of computational examples we demonstrate the capabilities of the REDLOG package for automatic offsetting and blending of solids that are commonly encountered in computational solid geometry. The present limitations of our method are due to the degree restrictions and the possible size of the output description.

## 2 Basic Definitions and the Tarski Principle

A *basic semialgebraic set* is of the form $\{\, \mathbf{x} \in \mathbb{R}^n \mid f(\mathbf{x}) \geqslant 0 \,\}$ for $f \in \mathbb{R}[\mathbf{x}]$. An arbitrary *semialgebraic set* in $\mathbb{R}^n$ is a Boolean combination of basic semialgebraic sets. For our practical computations we will restrict our attention to semialgebraic sets in $\mathbb{R}^n$ with defining polynomials $f \in \mathbb{Q}[\mathbf{x}]$ over the rational numbers. The following fact, due to A. Tarski [20], is fundamental:

[1] REDLOG is freely available on `http://www.fmi.uni-passau.de/~redlog/`.

*Let $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ denote the projection along the last coordinate. Whenever $A \subseteq \mathbb{R}^n$ is semialgebraic, then the image $\pi(A)$ is semialgebraic in $\mathbb{R}^{n-1}$.*

From the viewpoint of logic, this fact can be rephrased as quantifier elimination for the elementary theory of the reals: *Atomic formulas* are polynomial equations $f = 0$ and polynomial inequalities $f \geqslant 0$, $f > 0$ where $f \in \mathbb{Q}[\mathbf{x}]$. *Quantifier-free formulas* are obtained from atomic formulas by the logical operators "$\wedge$" (and), "$\vee$" (or), and "$\neg$" (not). Arbitrary *formulas* are obtained by these operations together with existential quantification "$\exists x$" (there exists an $x$) and universal quantification "$\forall x$" (for all $x$) over real variables. Any universal quantification $\forall x \varphi$ can be rewritten as an existential quantification $\neg \exists x \neg \varphi$. Existential quantification, in turn, corresponds geometrically to a projection. Consequently, Tarski's principle can be rephrased as follows:

*Every first-order formula in the theory of the reals is equivalent to a quantifier-free formula.*

So for every formula $\varphi(\mathbf{x})$ the set $S(\varphi) = \{\, \mathbf{a} \in \mathbb{R}^n \mid \varphi(\mathbf{a}) \text{ holds in } \mathbb{R} \,\}$ is a semialgebraic set. We call $\varphi$ a *description* of $S(\varphi)$. Accordingly, a quantifier-free formula $\varphi'$ is called a *quantifier-free description* of $S(\varphi')$.

Although our formulas are over the real numbers, we allow ourselves to write down constraints involving vectors. Consequently, quantifications such as "$\exists \mathbf{x}$" have to be read as "$\exists x_1 \dots \exists x_n$."

For formally writing down formulas involving applications of a metric, we have to ensure that the graph of this metric is semialgebraic set. Luckily, all metrics commonly encountered have this property. Throughout this paper we only use the Euclidean metric denoted by $d$. For points $\mathbf{x} \in \mathbb{R}^n$ and sets $A \subseteq \mathbb{R}^n$, we define as usual the *distance* of $\mathbf{x}$ to $A$ as $d(\mathbf{x}, A) = \inf_{\mathbf{a} \in A} d(\mathbf{x}, \mathbf{a})$, where for closed $A$ the infimum amounts to a minimum.

We denote by $\mathrm{B}(\mathbf{a}, r)$ and $\mathrm{B}[\mathbf{a}, r]$ the *open* and *closed ball*, respectively, with center $\mathbf{a} \in \mathbb{R}^n$ and radius $0 \leqslant r \in \mathbb{R}$. Balls are semialgebraic, and so are all objects commonly considered in solid modeling, in particular cylinders, cuboids, polyhedrons.

## 3 Solids as Regular Closed Semialgebraic Sets

Our intuition of a solid in $\mathbb{R}^3$ is that it is a closed semialgebraic set that is "everywhere 3-dimensional." It is thus reasonable and compatible with the more specialized definitions commonly used [13] to define a solid as a non-empty *regular closed* semialgebraic subset of $\mathbb{R}^3$.

For $A \subseteq \mathbb{R}^n$ the *regular closure* $A^*$ is defined as the closure of the interior $\overline{A^\circ}$ of $A$, and $A$ is called regular closed if $A^* = A$. The dual concept of regular open sets has been studied intensively in the theory of Boolean algebras, where it is the crucial tool for the completion of Boolean algebras [9]. The results on regular open sets can easily be transfered to regular closed sets.

For open sets $A$ we have $A^* \supseteq A$, and for closed sets $A$ we have $A^* \subseteq A$. If $A$ is neither open nor closed, there need not hold any inclusion, e.g.:

$$(]0,1[ \cup \{2\})^* = [0,1].$$

Since both interior and closure are monotone, so is the regular closure, i.e., $X \subseteq A$ implies $X^* \subseteq A^*$. It follows that if $X \subseteq A$ is already regular closed, then $X \subseteq A^*$. Finally, it is not hard to see that the regular closure is idempotent, i.e. $A^{**} = A^*$.

For closed sets $A$ the formation of $A^*$ will—roughly speaking—kill lower dimensional parts of $A$. This is exactly what is intended in the well-known formation of the normalized Boolean operations such as the normalized intersection: $A \cap^* B := (A \cap B)^*$. In the literature this regularization is usually carried out with all Boolean operations. From a computational point of view, it might be noteworthy that this is a certain overkill. For regular closed input solids $A$ and $B$ we have:

$$(A \cup B)^* = A \cup B, \quad (\sim A)^* = \overline{\sim A}, \quad (A \setminus B)^* = \overline{A \setminus B},$$

i.e., the union requires no regularization at all, and for regularizing complement and difference one need not take the interior. Note that, at least in our framework, it cannot be efficiently recognized whether a set is already open. So this external knowledge will save superfluous computations.

## 4 Boolean and Topological Operations on Solids

Taking unions, intersections, complements, interiors, and closures leads us from semialgebraic sets to semialgebraic sets:

$$\begin{aligned}
S(\varphi) \cup S(\psi) &= S(\varphi \vee \psi) \\
S(\varphi) \cap S(\psi) &= S(\varphi \wedge \psi) \\
\sim S(\varphi) &= S(\neg\varphi) \\
S(\varphi(\mathbf{x}))^\circ &= S(\exists r(r > 0 \wedge \forall \mathbf{u}(d(\mathbf{x},\mathbf{u}) < r \longrightarrow \varphi(\mathbf{u})))) \\
\overline{S(\varphi(\mathbf{x}))} &= S(\forall r(r > 0 \longrightarrow \exists \mathbf{u}(d(\mathbf{x},\mathbf{u}) < r \wedge \varphi(\mathbf{u})))).
\end{aligned}$$

So does any composition of these, such as the regular closure and regularized Boolean operations or the boundary $\partial S(\varphi) = \overline{S(\varphi)} \cap \overline{\sim S(\varphi)}$.

Note that in the definitions of the interior and the closure, the new variables, $r$ and the coordinates of $\mathbf{u}$, occur only bound by quantifiers. An equivalent quantifier-free description will, such as the input description $\varphi$, involve only the coordinates $x_1, \ldots, x_n$ of $\mathbf{x}$.

## 5 Offsets

An important geometrical construction for our purposes is the formation of *offsets*. Given *arbitrary* $A \subseteq \mathbb{R}^n$ and $0 < r \in \mathbb{R}$ we define:

$$\mathrm{off}_{\leqslant \mathrm{r}}(A) = \{\, \mathbf{x} \in \mathbb{R}^n \mid \exists(\mathbf{a} \in A) : d(\mathbf{x}, \mathbf{a}) \leqslant r \,\}.$$

The offset of a semialgebraic set is again semialgebraic:

$$\mathrm{off}_{\leqslant \mathrm{r}}(S(\varphi(\mathbf{x}))) = S(\exists \mathbf{u}(\varphi(\mathbf{u}) \wedge d(\mathbf{x}, \mathbf{u}) \leqslant r)).$$

This notion of an offset, also called *expanding*, is quite standard [17]. Also, it has already been applied to non-regular sets [16]. For regular closed $A$, we have that $\mathrm{off}_{\leqslant \mathrm{r}}(A)$ is again regular closed.

There is an "opposite" operation of *shrinking* a set. The result of this is not necessarily regular closed even when applied to regular closed input sets. Thus shrinking has usually been used in combination with regularization similar to the regularized Boolean operations discussed above. Compositions of expanding and shrinking have been used to perform blending operations on solids [17].

Combining expanding and shrinking for blending solids, one is unfortunately faced with unintuitive results, (cf. Section 7). We will exhibit that these problems are actually caused by the regularization of shrinkings. To overcome the problems, we choose an alternate approach to defining blendings in terms of offsets. Therefore, we generalize our notion of an offset by defining also the following:

$$\begin{aligned}
\mathrm{off}_{\geqslant \mathrm{r}}(A) &= \{\, \mathbf{x} \in \mathbb{R}^n \mid \forall(\mathbf{a} \in A) : d(\mathbf{x}, \mathbf{a}) \geqslant r \,\} \\
\mathrm{off}_{< \mathrm{r}}(A) &= \{\, \mathbf{x} \in \mathbb{R}^n \mid \exists(\mathbf{a} \in A) : d(\mathbf{x}, \mathbf{a}) < r \,\} \\
\mathrm{off}_{> \mathrm{r}}(A) &= \{\, \mathbf{x} \in \mathbb{R}^n \mid \forall(\mathbf{a} \in A) : d(\mathbf{x}, \mathbf{a}) > r \,\}
\end{aligned}$$

Just like $\mathrm{off}_{\leqslant \mathrm{r}}$, these operations lead from semialgebraic sets to semialgebraic sets, e.g.,

$$\mathrm{off}_{\geqslant \mathrm{r}}(S(\varphi(\mathbf{x}))) = S(\forall \mathbf{u}(\varphi(\mathbf{u}) \longrightarrow d(\mathbf{x},\mathbf{u}) \leqslant r)).$$

The rest of this section is devoted to a careful study of the properties of our various offsets and their interaction with Boolean and topological operations.

Our definitions are not necessarily conform with the notion of *distance* between points and sets. For instance $1 \notin \mathrm{off}_{\leqslant 1}\,]-\infty, 0[\, = \,]-\infty, 1[$, but

$$d(1, ]-\infty, 0[) = \inf\{\, d(1, x) \mid x < 0 \,\} = 1.$$

In other words, there is a point of distance 1 from the given set not contained in its $\mathrm{off}_{\leqslant 1}$. For closed $A \subseteq \mathbb{R}^n$, we can, however, assert that

$$\mathrm{off}_{\leqslant \mathrm{r}}(A) = \{\, \mathbf{x} \in \mathbb{R}^n \mid d(\mathbf{x}, A) \leqslant r \,\}$$
$$\mathrm{off}_{\geqslant \mathrm{r}}(A) = \{\, \mathbf{x} \in \mathbb{R}^n \mid d(\mathbf{x}, A) \geqslant r \,\}.$$

It follows immediately from the definitions that $\mathrm{off}_{\leqslant \mathrm{r}}(A) = \sim \mathrm{off}_{> \mathrm{r}}(A)$ and $\mathrm{off}_{< \mathrm{r}}(A) = \sim \mathrm{off}_{\geqslant \mathrm{r}}(A)$. The following lemma is concerned with offsets of unions.

**Lemma 1** *For $A \subseteq \mathbb{R}^n$ and $0 < r \in \mathbb{R}$ the following assertions hold:*

*(1)* $\mathrm{off}_{\leqslant \mathrm{r}}(A \cup B) = \mathrm{off}_{\leqslant \mathrm{r}}(A) \cup \mathrm{off}_{\leqslant \mathrm{r}}(B)$
*(2)* $\mathrm{off}_{\geqslant \mathrm{r}}(A \cup B) = \mathrm{off}_{\geqslant \mathrm{r}}(A) \cap \mathrm{off}_{\geqslant \mathrm{r}}(B)$.

*Analogous results hold when replacing "$\leqslant$" by "$<$" and "$\geqslant$" by "$>$," respectively.*

**PROOF.**

(1) $\mathbf{x} \in \mathrm{off}_{\leqslant \mathrm{r}}(A \cup B)$ iff there exists $\mathbf{y} \in A \cup B$ such that $d(\mathbf{x},\mathbf{y}) \leqslant r$ iff there exists $\mathbf{y} \in A$ such that $d(\mathbf{x},\mathbf{y}) \leqslant r$ or there exists $\mathbf{y} \in B$ such that $d(\mathbf{x},\mathbf{y}) \leqslant r$ iff $\mathbf{x} \in \mathrm{off}_{\leqslant \mathrm{r}}(A)$ or $\mathbf{x} \in \mathrm{off}_{\leqslant \mathrm{r}}(B)$ iff $\mathbf{x} \in \mathrm{off}_{\leqslant \mathrm{r}}(A) \cup \mathrm{off}_{\leqslant \mathrm{r}}(B)$. The same argument holds for "$<$" instead of "$\leqslant$."
(2) $\mathbf{x} \in \mathrm{off}_{\geqslant \mathrm{r}}(A \cup B)$ iff for all $\mathbf{c} \in A \cup B$ we have $d(\mathbf{x},\mathbf{c}) \geqslant r$ iff for all $\mathbf{a} \in A$ we have $d(\mathbf{x},\mathbf{a}) \geqslant r$ and for all $\mathbf{b} \in B$ we have $d(\mathbf{x},\mathbf{b}) \geqslant r$ iff $\mathbf{x} \in \mathrm{off}_{\geqslant \mathrm{r}}(A)$ and $\mathbf{x} \in \mathrm{off}_{\geqslant \mathrm{r}}(B)$ iff $\mathbf{x} \in \mathrm{off}_{\geqslant \mathrm{r}}(A) \cap \mathrm{off}_{\geqslant \mathrm{r}}(B)$. The same argument holds for "$>$" instead of "$\geqslant$." □

Next, the closure properties of our offsets are quite surprising. For some offsets they depend on the set considered, while for other offsets they do not so.

**Lemma 2** *For $A \subseteq \mathbb{R}^n$ and $0 < r \in \mathbb{R}$ the following assertions hold:*

*(1)* $\mathrm{off}_{<r}(A)$ *is open and* $\mathrm{off}_{\geqslant r}(A)$ *is closed.*
*(2) If $A$ is open, then* $\mathrm{off}_{\leqslant r}(A)$ *is open and* $\mathrm{off}_{>r}(A)$ *is closed.*
*(3) If $A$ is closed, then* $\mathrm{off}_{\leqslant r}(A)$ *is closed and* $\mathrm{off}_{>r}(A)$ *is open.*

**PROOF.**

(1) Let $\mathbf{x} \in \mathrm{off}_{<r}(A)$, pick $\mathbf{a} \in A$, $0 < \varepsilon \in \mathbb{R}$ such that $d(\mathbf{x}, \mathbf{a}) = r - \varepsilon < r$. For $\mathbf{y} \in \mathrm{B}(\mathbf{x}, \varepsilon)$ we have $d(\mathbf{y}, \mathbf{a}) \leqslant d(\mathbf{y}, \mathbf{x}) + d(\mathbf{x}, \mathbf{a}) < \varepsilon + r - \varepsilon = r$, i.e., $\mathrm{B}(\mathbf{x}, \varepsilon) \subseteq \mathrm{off}_{<r}(A)$.
(2) For open $A$ let $\mathbf{x} \in \mathrm{off}_{\leqslant r}(A)$, say $\mathbf{a} \in A$ such that $d(\mathbf{x}, \mathbf{a}) \leqslant r$. Choose $0 < \varepsilon \in \mathbb{R}$ such that $\mathrm{B}(\mathbf{a}, \varepsilon) \subseteq A$. For $\mathbf{y} \in \mathrm{B}(\mathbf{x}, \frac{\varepsilon}{2})$ pick $\mathbf{z}$ on the line segment $\overline{\mathbf{ax}}$ with $d(\mathbf{a}, \mathbf{z}) = \frac{\varepsilon}{2}$. Then $d(\mathbf{y}, \mathbf{z}) \leqslant d(\mathbf{y}, \mathbf{x}) + d(\mathbf{x}, \mathbf{z}) < \frac{\varepsilon}{2} + r - \frac{\varepsilon}{2} = r$, i.e., $\mathrm{B}(\mathbf{x}, \frac{\varepsilon}{2}) \subseteq \mathrm{off}_{\leqslant r}(A)$.
(3) For closed $A$, we show that $\overline{\mathrm{off}_{\leqslant r}(A)} \subseteq \mathrm{off}_{\leqslant r}(A)$: Let $\mathbf{x} \in \overline{\mathrm{off}_{\leqslant r}(A)}$, and assume for a contradiction that $\mathbf{x} \in \mathrm{off}_{>r}(A) = {\sim}\mathrm{off}_{\leqslant r}(A)$. Since $A$ is closed, there is $\mathbf{a} \in A$ and $0 < \varepsilon \in \mathbb{R}$ such that $d(\mathbf{x}, \mathbf{a}) = d(\mathbf{x}, A) = r + \varepsilon > r$. Pick $\mathbf{y} \in \mathrm{B}(\mathbf{x}, \varepsilon) \cap \mathrm{off}_{\leqslant r}(A) \neq \emptyset$, then $d(\mathbf{x}, \mathbf{a}) \leqslant d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{a}) < \varepsilon + r$, a contradiction. $\square$

Finally, we turn to the interplay between offsets and topological operations and regularizations.

**Lemma 3** *(1) Let $A \subseteq \mathbb{R}^n$, and let $0 < r \in \mathbb{R}$. Then*

$$\begin{aligned}
\mathrm{off}_{>r}(A)^\circ &= \overset{\circ}{\overline{\mathrm{off}_{>r}(A)}} = \mathrm{off}_{\geqslant r}(A)^\circ = \overset{\circ}{\overline{\mathrm{off}_{\geqslant r}(A)}} = \mathrm{off}_{>r}(\overline{A}) \\
&\subseteq \mathrm{off}_{>r}(A) \\
&\subseteq \overline{\mathrm{off}_{>r}(A)} \\
&\subseteq \mathrm{off}_{\geqslant r}(\overline{A}) = \mathrm{off}_{\geqslant r}(A) = \overline{\mathrm{off}_{\geqslant r}(A)}, \\
\mathrm{off}_{>r}(A)^* &= \mathrm{off}_{\geqslant r}(A)^*.
\end{aligned}$$

*(2) If $A$ is open, then*

$$\begin{aligned}
\mathrm{off}_{>r}(A)^\circ &= \overset{\circ}{\overline{\mathrm{off}_{>r}(A)}} = \mathrm{off}_{\geqslant r}(A)^\circ = \overset{\circ}{\overline{\mathrm{off}_{\geqslant r}(A)}} = \mathrm{off}_{>r}(\overline{A}) \\
&\subseteq \mathrm{off}_{>r}(A)^* = \mathrm{off}_{\geqslant r}(A)^* \\
&\subseteq \mathrm{off}_{>r}(A) = \overline{\mathrm{off}_{>r}(A)} = \mathrm{off}_{\geqslant r}(\overline{A}) = \mathrm{off}_{\geqslant r}(A) = \overline{\mathrm{off}_{\geqslant r}(A)}.
\end{aligned}$$

*(3) If $A$ is closed, then*

$$\mathrm{off}_{>r}(A)^\circ = \overset{\circ}{\overline{\mathrm{off}_{>r}(A)}} = \mathrm{off}_{\geqslant r}(A)^\circ = \overset{\circ}{\overline{\mathrm{off}_{\geqslant r}(A)}} = \mathrm{off}_{>r}(\overline{A}) = \mathrm{off}_{>r}(A)$$

$$\subseteq \mathrm{off}_{>\mathrm{r}}(A)^* = \overline{\mathrm{off}_{>\mathrm{r}}(A)} = \mathrm{off}_{\geqslant \mathrm{r}}(A)^*$$
$$\subseteq \mathrm{off}_{\geqslant \mathrm{r}}(\overline{A}) = \mathrm{off}_{\geqslant \mathrm{r}}(A) = \overline{\mathrm{off}_{\geqslant \mathrm{r}}(A)}.$$

**PROOF.**

(1) The inclusion $\mathrm{off}_{>\mathrm{r}}(A)^\circ \subseteq \mathring{\overline{\mathrm{off}_{>\mathrm{r}}(A)}}$ follows from $\mathrm{off}_{>\mathrm{r}}(A) \subseteq \overline{\mathrm{off}_{>\mathrm{r}}(A)}$ by the monotony of the interior operator.

In the same way $\mathring{\overline{\mathrm{off}_{>\mathrm{r}}(A)}} \subseteq \mathrm{off}_{\geqslant \mathrm{r}}(A)^\circ$ follows from $\overline{\mathrm{off}_{>\mathrm{r}}(A)} \subseteq \mathrm{off}_{\geqslant \mathrm{r}}(A)$ shown below.

Since $\mathrm{off}_{\geqslant \mathrm{r}}(A)$ is closed by Lemma 2, we have $\mathrm{off}_{\geqslant \mathrm{r}}(A)^\circ = \mathring{\overline{\mathrm{off}_{\geqslant \mathrm{r}}(A)}}$.

We show $\mathrm{off}_{\geqslant \mathrm{r}}(A)^\circ \subseteq \mathrm{off}_{>\mathrm{r}}(\overline{A})$: Let $\mathbf{x} \in \mathrm{off}_{\geqslant \mathrm{r}}(A)^\circ$, and let $\mathbf{y} \in \overline{A}$. Assume for a contradiction that $d(\mathbf{x}, \mathbf{y}) \leqslant r$. Choose $0 < \varepsilon \in \mathbb{R}$ such that $\mathrm{B}(\mathbf{x}, \varepsilon) \subseteq \mathrm{off}_{\geqslant \mathrm{r}}(A)$. Pick $\mathbf{z}$ on the line segment $\overline{\mathbf{xy}}$ such that $d(\mathbf{x}, \mathbf{z}) = \frac{\varepsilon}{2}$, and pick $\mathbf{a} \in \mathrm{B}(\mathbf{y}, \frac{\varepsilon}{2}) \cap A$. Then $d(\mathbf{z}, \mathbf{a}) \leqslant d(\mathbf{z}, \mathbf{y}) + d(\mathbf{y}, \mathbf{a}) < r - \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = r$, but $\mathbf{z} \in \mathrm{off}_{\geqslant \mathrm{r}}(A)$, a contradiction.

$\mathrm{off}_{>\mathrm{r}}(\overline{A}) \subseteq \mathrm{off}_{>\mathrm{r}}(A) \subseteq \overline{\mathrm{off}_{>\mathrm{r}}(A)}$ is obvious.

We show the contrapositive of $\mathrm{off}_{>\mathrm{r}}(A) \subseteq \mathrm{off}_{\geqslant \mathrm{r}}(\overline{A})$. This implies that $\overline{\mathrm{off}_{>\mathrm{r}}(A)} \subseteq \mathrm{off}_{\geqslant \mathrm{r}}(\overline{A})$ since $\mathrm{off}_{\geqslant \mathrm{r}}(\overline{A})$ is closed: Let $\mathbf{x} \notin \mathrm{off}_{\geqslant \mathrm{r}}(\overline{A})$, say $\mathbf{y} \in \overline{A}$ and $0 < \varepsilon \in \mathbb{R}$ such that $d(\mathbf{x}, \mathbf{y}) = r - \varepsilon < r$. Choose $\mathbf{a} \in \mathrm{B}(\mathbf{y}, \varepsilon) \cap A$. Then $d(\mathbf{x}, \mathbf{a}) \leqslant d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{a}) < r - \varepsilon + \varepsilon = r$, i.e., $\mathbf{x} \notin \mathrm{off}_{>\mathrm{r}}(A)$.

$\mathrm{off}_{\geqslant \mathrm{r}}(\overline{A}) \subseteq \mathrm{off}_{\geqslant \mathrm{r}}(A) \subseteq \overline{\mathrm{off}_{\geqslant \mathrm{r}}(A)}$ is again obvious.

We know by Lemma 2 that $\mathrm{off}_{>\mathrm{r}}(\overline{A}) = \mathrm{off}_{>\mathrm{r}}(\overline{A})^\circ$. From $\mathrm{off}_{>\mathrm{r}}(\overline{A}) \subseteq \mathrm{off}_{>\mathrm{r}}(A)$ proved above, it follows by the monotony of the interior operator that $\mathrm{off}_{>\mathrm{r}}(\overline{A})^\circ \subseteq \mathrm{off}_{>\mathrm{r}}(A)^\circ$; together $\mathrm{off}_{>\mathrm{r}}(\overline{A}) \subseteq \mathrm{off}_{>\mathrm{r}}(A)^\circ$.

We show the contrapositive of $\mathrm{off}_{\geqslant \mathrm{r}}(A) \subseteq \mathrm{off}_{\geqslant \mathrm{r}}(\overline{A})$: Let $\mathbf{x} \notin \mathrm{off}_{\geqslant \mathrm{r}}(\overline{A})$, say $\mathbf{y} \in \overline{A}$, $0 < \varepsilon \in \mathbb{R}$ such that $d(\mathbf{x}, \mathbf{y}) = r - \varepsilon < r$. Then for $\mathbf{a} \in \mathrm{B}(\mathbf{y}, \varepsilon) \cap A$ we have $d(\mathbf{x}, \mathbf{a}) \leqslant d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{a}) < r - \varepsilon + \varepsilon = r$, i.e., $\mathbf{x} \notin \mathrm{off}_{\geqslant \mathrm{r}}(A)$.

We know that $\mathrm{off}_{\geqslant \mathrm{r}}(A) = \overline{\mathrm{off}_{\geqslant \mathrm{r}}(A)}$ by Lemma 2.

From $\mathrm{off}_{>\mathrm{r}}(A)^\circ = \mathrm{off}_{\geqslant \mathrm{r}}(A)^\circ$ it follows that $\mathrm{off}_{>\mathrm{r}}(A)^* = \mathrm{off}_{\geqslant \mathrm{r}}(A)^*$.

(2) $\mathrm{off}_{>\mathrm{r}}(A)^\circ \subseteq \mathrm{off}_{>\mathrm{r}}(A)^*$ is obvious.

Since $\mathrm{off}_{>\mathrm{r}}(A)$ is closed by Lemma 2, we may conclude that $\mathrm{off}_{>\mathrm{r}}(A)^* \subseteq \mathrm{off}_{>\mathrm{r}}(A)$.

We show that $\mathrm{off}_{\geqslant \mathrm{r}}(A) \subseteq \mathrm{off}_{>\mathrm{r}}(A)$: Let $\mathbf{x} \in \mathrm{off}_{\geqslant \mathrm{r}}(A)$, and let $\mathbf{a} \in A$. Then $d(\mathbf{x}, \mathbf{a}) \geqslant r$. Assume for a contradiction that $d(\mathbf{x}, \mathbf{a}) = r$. Pick $0 < \varepsilon \in \mathbb{R}$ such that $\mathrm{B}(\mathbf{a}, \varepsilon) \subseteq A$. Let $\mathbf{z}$ be a point on the line segment $\overline{\mathbf{xa}}$ with $d(\mathbf{z}, \mathbf{a}) = \frac{\varepsilon}{2}$. Then $\mathbf{z} \in A$ but $d(\mathbf{x}, \mathbf{z}) = r - \frac{\varepsilon}{2} < r$, a contradiction.

(3) If $A$ is already closed, then obviously $\mathrm{off}_{>\mathrm{r}}(\overline{A}) = \mathrm{off}_{>\mathrm{r}}(A)$.

We know by Lemma 2 that $\mathrm{off}_{>\mathrm{r}}(A)$ is open and thus $\mathrm{off}_{>\mathrm{r}}(A)^* = \overline{\mathrm{off}_{>\mathrm{r}}(A)}$. □

A similar result for $<$-offsets and $\leqslant$-offsets is derived in the following way: We take the complement of each offset in Lemma 3, which inverts all inclusion relations. Then the complements are encoded into offsets using rules as discussed above together with the topological rules $\sim \overline{X} = (\sim X)^\circ$ and $\sim (X^\circ) = \overline{\sim X}$:

**Corollary 4** *(1) Let* $A \subseteq \mathbb{R}^n$*, and let* $0 < r \in \mathbb{R}$*. Then*

$$\begin{aligned}
\mathrm{off}_{<\mathrm{r}}(A)^\circ &= \mathrm{off}_{<\mathrm{r}}(A) = \mathrm{off}_{<\mathrm{r}}(\overline{A}) \\
&\subseteq \mathrm{off}_{\leqslant\mathrm{r}}(A)^\circ \\
&\subseteq \mathrm{off}_{\leqslant\mathrm{r}}(A) \\
&\subseteq \mathrm{off}_{\leqslant\mathrm{r}}(\overline{A}) = \mathrm{off}_{<\mathrm{r}}(A)^* = \overline{\mathrm{off}_{<\mathrm{r}}(A)} = \mathrm{off}_{\leqslant\mathrm{r}}(A)^* = \overline{\mathrm{off}_{\leqslant\mathrm{r}}(A)}, \\
\overset{\circ}{\overline{\mathrm{off}_{<\mathrm{r}}(A)}} &= \overset{\circ}{\overline{\mathrm{off}_{\leqslant\mathrm{r}}(A)}}.
\end{aligned}$$

*(2) If* $A$ *is open, then*

$$\begin{aligned}
\mathrm{off}_{<\mathrm{r}}(A)^\circ &= \mathrm{off}_{<\mathrm{r}}(A) = \mathrm{off}_{<\mathrm{r}}(\overline{A}) = \mathrm{off}_{\leqslant\mathrm{r}}(A)^\circ = \mathrm{off}_{\leqslant\mathrm{r}}(A) \\
&\subseteq \overset{\circ}{\overline{\mathrm{off}_{<\mathrm{r}}(A)}} = \overset{\circ}{\overline{\mathrm{off}_{\leqslant\mathrm{r}}(A)}} \\
&\subseteq \mathrm{off}_{\leqslant\mathrm{r}}(\overline{A}) = \mathrm{off}_{<\mathrm{r}}(A)^* = \overline{\mathrm{off}_{<\mathrm{r}}(A)} = \mathrm{off}_{\leqslant\mathrm{r}}(A)^* = \overline{\mathrm{off}_{\leqslant\mathrm{r}}(A)}.
\end{aligned}$$

*(3) If* $A$ *is closed, then*

$$\begin{aligned}
\mathrm{off}_{<\mathrm{r}}(A)^\circ &= \mathrm{off}_{<\mathrm{r}}(A) = \mathrm{off}_{<\mathrm{r}}(\overline{A}) \\
&\subseteq \overset{\circ}{\overline{\mathrm{off}_{<\mathrm{r}}(A)}} = \mathrm{off}_{\leqslant\mathrm{r}}(A)^\circ = \overset{\circ}{\overline{\mathrm{off}_{\leqslant\mathrm{r}}(A)}} \\
&\subseteq \mathrm{off}_{\leqslant\mathrm{r}}(A) = \mathrm{off}_{\leqslant\mathrm{r}}(\overline{A}) = \mathrm{off}_{<\mathrm{r}}(A)^* = \overline{\mathrm{off}_{<\mathrm{r}}(A)} = \mathrm{off}_{\leqslant\mathrm{r}}(A)^* \\
&= \overline{\mathrm{off}_{\leqslant\mathrm{r}}(A)}.
\end{aligned}$$

The following table illustrates by example that already in $\mathbb{R}$ all inclusions in Lemma 3 are proper in general. The examples can be transferred to $</\leqslant$-offsets. Replacing the intervals by corresponding balls, they can be lifted to any dimension.

| $A$ | $\mathrm{off}_{>1}(\overline{A})$ | $\mathrm{off}_{>1}(A)^*$ | $\mathrm{off}_{>1}(A)$ | $\overline{\mathrm{off}_{>1}(A)}$ | $\mathrm{off}_{\geqslant 1}(\overline{A})$ |
|---|---|---|---|---|---|
| $]-\infty, 0[$ | $]1, \infty[$ | $[1, \infty[$ | | | |
| $\sim [-1, 1]$ | | $\emptyset$ | $\{0\}$ | | |
| $]-\infty, 0]$ | | | $]1, \infty[$ | $[1, \infty[$ | |
| $\sim ]-1, 1[$ | | | | $\emptyset$ | $\{0\}$ |

Since all the inclusions are proper, it follows that $\mathrm{off}_{>\mathrm{r}}(A)^*$ cannot be positioned into the given chain of inclusions for the general case. The following example illustrates this fact once more: We set

$$A = ]-\infty, 0] \cup [3, 4[ \cup ]6, \infty[\,.$$

Then $\mathrm{off}_{>1}(A)^* = [1, 2]$ is neither subset nor superset to $\mathrm{off}_{>1}(A) = ]1, 2[ \cup \{5\}$. The same observation applies to $\mathrm{off}_{\geqslant \mathrm{r}}(A)^* = \mathrm{off}_{>\mathrm{r}}(A)^*$.

## 6 Blendings

Among the numerous types of blendings of solids in the literature [11,21] we consider global fixed-radius rolling ball roundings in the classification of Vida et al. [21]

Our blending operations are by definition *global* on the involved objects. In practice, however, one often wishes to formate *local* blendings, i.e., blendings restricted to certain edges or corners. This can be achieved by equivalently modifying the Boolean structure of the quantifier-free representation of a given solid in such a way that the target parts are isolated. Then the operations are applied only to the subformulas describing these parts. Afterwards, the Boolean structure can once more be rearranged to a more canonical form.

### *6.1 Roundings from Inside*

Let $A$ be a closed set in $\mathbb{R}^n$, and let $0 \leqslant r \in \mathbb{R}$. Then the *$r$-rounding from inside* of $A$ is the union of all closed balls completely contained in $A$:

$$\mathrm{round}_r(A) = \bigcup_{\mathrm{B}[\mathbf{a},r] \subseteq A} \mathrm{B}[\mathbf{a}, r] = \{\, \mathbf{x} \in \mathbb{R}^n \mid \exists(\mathbf{a} \in A) : \mathbf{x} \in \mathrm{B}[\mathbf{a}, r] \subseteq A \,\}.$$

Accordingly, $A$ is called *$r$-rounded from inside* if $A = \mathrm{round}_r(A)$.

From the definition it follows that $\mathrm{round}_r(A) \subseteq A$, and for $A \subseteq B$ we may conclude that $\mathrm{round}_r(A) \subseteq \mathrm{round}_r(B)$. The $r$-rounding from inside of a semialgebraic set is again semialgebraic:

$$\mathrm{round}_r(S(\varphi(\mathbf{x}))) = S(\exists \mathbf{a}(d(\mathbf{a}, \mathbf{x}) \leqslant r \wedge \forall \mathbf{y}(d(\mathbf{a}, \mathbf{y}) \leqslant r \longrightarrow \varphi(\mathbf{y})))).$$

The following theorem characterizes roundings from inside in terms of offsets. From this characterization, it follows that $r$-roundings from inside are, though *infinite* unions of closed balls, again closed. In fact they are even regular closed.

**Theorem 5** *Let $A \subseteq \mathbb{R}^n$ be closed, and let $0 \leqslant r \in \mathbb{R}$. Then*

$$\mathrm{round}_r(A) = \mathrm{off}_{\leqslant \mathrm{r}}\mathrm{off}_{>\mathrm{r}}(\sim A) = \mathrm{off}_{\leqslant \mathrm{r}}\mathrm{off}_{\geqslant \mathrm{r}}(\overline{\sim A}).$$

*In particular* $\mathrm{round}_r(A)$ *is regular closed.*

**PROOF.** Let $\mathbf{x} \in \mathrm{round}_r(A)$, and pick $\mathbf{y} \in A$ with $\mathbf{x} \in \mathrm{B}[\mathbf{y}, r] \subseteq A$. Then $\mathbf{y} \in \mathrm{off}_{>\mathrm{r}}(\sim A)$ and $d(\mathbf{x}, \mathbf{y}) \leqslant r$, and so $\mathbf{x} \in \mathrm{off}_{\leqslant \mathrm{r}}\mathrm{off}_{>\mathrm{r}}(\sim A)$.

Conversely, let $\mathbf{x} \in \mathrm{off}_{\leqslant \mathrm{r}}\mathrm{off}_{>\mathrm{r}}(\sim A)$ and pick $\mathbf{y} \in \mathrm{off}_{>\mathrm{r}}(\sim A)$ with $d(\mathbf{x}, \mathbf{y}) \leqslant r$. Then $\mathbf{x} \in \mathrm{B}[\mathbf{y}, r] \subseteq A$, i.e. $\mathbf{x} \in \mathrm{round}_r(A)$.

The rest follows by Lemma 3 and Corollary 4. $\square$

Let $A \subseteq \mathbb{R}^n$ be closed. We obviously have $\mathrm{round}_0(A) = A$. For $0 < r \in \mathbb{R}$ it follows immediately from Theorem 5 that $\mathrm{round}_r(A) \subseteq A^*$, and further

$$\bigcup_{0<r\in\mathbb{R}} \mathrm{round}_r(A) \subseteq A^*.$$

This inclusion relation is in general a proper one: Consider the first quadrant $A = \{ (x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geqslant 0 \text{ and } x_2 \geqslant 0 \}$ of real 2-space. It is easily seen to be regular closed, but for all $0 < r \in \mathbb{R}$ the origin $(0, 0)$ is not in $\mathrm{round}_r(A)$.

We eventually have to clarify whether $r$-roundings are actually $r$-rounded. They are, and moreover they are as close to the original set as possible:

**Lemma 6** *Let $A \subseteq \mathbb{R}^n$ be closed, and let $0 \leqslant r \in \mathbb{R}$. Then* $\mathrm{round}_r(A)$ *is the largest subset of $A$ that is $r$-rounded.*

**PROOF.** We are allowed to form $\mathrm{round}_r(\mathrm{round}_r(A))$, because $\mathrm{round}_r(A)$ is closed by Theorem 5. Since $\mathrm{round}_r(A)$ itself is already defined as union of closed $r$-balls, it follows that

$$\mathrm{round}_r(\mathrm{round}_r(A)) = \bigcup \{ \mathrm{B}[\mathbf{b}, r] \mid \mathrm{B}[\mathbf{b}, r] \subseteq \mathrm{round}_r(A) \} = \mathrm{round}_r(A),$$

i.e., $\mathrm{round}_r(A)$ is $r$-rounded from inside. Let $B \subseteq A$ be $r$-rounded from inside. Then $B = \mathrm{round}_r(B) \subseteq \mathrm{round}_r(A)$. $\square$

Finally, we examine the interplay between roundings from inside and Boolean operations.

**Lemma 7** *Let $A$, $B \subseteq \mathbb{R}^n$ be closed, and let $0 < r \in \mathbb{R}$. Then the following assertions hold:*

*(1)* $\mathrm{round}_r(A) \cup \mathrm{round}_r(B) \subseteq \mathrm{round}_r(A \cup B)$
*(2)* $\mathrm{round}_r(A) \cap \mathrm{round}_r(B) \supseteq \mathrm{round}_r(A \cap B)$
*(3) If both $A$ and $B$ are $r$-rounded from inside, then so is $A \cup B$.*

**PROOF.**

(1) Let $\mathbf{x} \in \mathrm{round}_r(A) \cup \mathrm{round}_r(B)$, say $\mathbf{x} \in \mathrm{round}_r(A)$. Then there exists $\mathbf{y} \in A$ such that $\mathbf{x} \in \mathrm{B}[\mathbf{y}, r] \subseteq A \subseteq A \cup B$.
(2) Let $\mathbf{x} \in \mathrm{round}_r(A \cap B)$. Then there exists $\mathbf{y} \in A \cap B$ with $\mathbf{x} \in \mathrm{B}[\mathbf{y}, r] \subseteq A \cap B \subseteq A, B$.
(3) Let $\mathbf{x} \in A \cup B$, say $\mathbf{x} \in A$. Then there exists $\mathbf{y} \in A$ with $\mathbf{x} \in \mathrm{B}[\mathbf{y}, r] \subseteq A \subseteq A \cup B$. $\square$

The inclusions in the Lemma are proper in general: Consider real 2-space and fix $r = 1$. For the first inclusion consider two adjacent squares of side lengths 2. For the second inclusion consider two overlapping closed unit disks centered at $(0, 0)$ and $(1, 0)$. This latter example shows also that the intersection of $r$-rounded sets need not be $r$-rounded.

### *6.2 Roundings from Outside*

Recall that we have defined roundings from inside of a solid as the union of all closed balls contained in it. This removes certain parts of the solid such as convex edges. Roundings from outside, in contrast, will add certain parts to the solid thus smoothing concave edges. Roughly speaking, the rounding from outside of a solid is obtained by rounding its complement from inside, i.e. as the complement of all balls not intersecting it. Note, however, that both the input solid and the result of the rounding operation should be closed, i.e. the corresponding complements are open. We hence operate with open balls here, and, as a consequence, the notion of rounding from outside is not exactly dual to that of rounding from inside.

Let $A$ be a closed set in $\mathbb{R}^n$, and let $0 \leqslant r \in \mathbb{R}$. Then the *$r$-rounding from outside* of $A$ is defined as the following set:

$$\begin{aligned}\mathrm{Round}_r(A) &= \bigcap \{\, \sim \mathrm{B}(\mathbf{x}, r) \mid \mathrm{B}(\mathbf{x}, r) \cap A = \emptyset \,\} \\ &= \{\, \mathbf{x} \in \mathbb{R}^n \mid \forall (\mathbf{y} \in \mathbb{R}^n) : \mathrm{B}(\mathbf{y}, r) \cap A = \emptyset \Longrightarrow \mathbf{x} \notin \mathrm{B}(\mathbf{y}, r) \,\}.\end{aligned}$$

A is called *r-rounded from outside* if $A = \mathrm{Round}_r(A)$.

From the definition it follows that $A \subseteq \mathrm{Round}_r(A)$, and for $A \subseteq B$, we can conclude that $\mathrm{Round}_r(A) \subseteq \mathrm{Round}_r(B)$. As an intersection of closed sets $\mathrm{Round}_r(A)$ is closed. Roundings from outside of semialgebraic sets are semi-algebraic:

$$\mathrm{Round}_r(S(\varphi(\mathbf{x}))) = S(\forall \mathbf{y}(\neg \exists \mathbf{z}(d(\mathbf{y}, \mathbf{z}) < r \wedge \varphi(\mathbf{z})) \longrightarrow d(\mathbf{y}, \mathbf{x}) \geqslant r)).$$

Similar to roundings from inside, roundings from outside can be characterized as iterated offset computations:

**Theorem 8** *Let* $A \subseteq \mathbb{R}^n$ *be closed, and let* $0 \leqslant r \in \mathbb{R}$. *Then* $\mathrm{Round}_r(A) = \mathrm{off}_{\geqslant \mathrm{r}} \mathrm{off}_{\geqslant \mathrm{r}}(A)$.

**PROOF.** Let $\mathbf{x} \in \mathrm{Round}_r(A)$, and let $\mathbf{y} \in \mathrm{off}_{\geqslant \mathrm{r}}(A)$. Then $\mathrm{B}(\mathbf{y}, r) \cap A = \emptyset$, thus $\mathbf{x} \notin \mathrm{B}(\mathbf{y}, r)$, and so $d(\mathbf{x}, \mathbf{y}) \geqslant r$. This shows that $\mathrm{Round}_r(A) \subseteq \mathrm{off}_{\geqslant \mathrm{r}} \mathrm{off}_{\geqslant \mathrm{r}}(A)$.

Conversely, let $\mathbf{x} \in \mathrm{off}_{\geqslant \mathrm{r}} \mathrm{off}_{\geqslant \mathrm{r}}(A)$, and let $\mathbf{y} \in \mathbb{R}^n$ be such that $\mathrm{B}(\mathbf{y}, r) \cap A = \emptyset$. Then $\mathbf{y} \in \mathrm{off}_{\geqslant \mathrm{r}}(A)$, thus $d(\mathbf{x}, \mathbf{y}) \geqslant r$, and so $\mathbf{x} \notin \mathrm{B}(\mathbf{y}, r)$. This shows the converse inclusion. □

Also, rounding from outside yields rounded sets as one would expect:

**Lemma 9** *Let* $A \subseteq \mathbb{R}^n$ *be closed, and let* $0 \leqslant r \in \mathbb{R}$. *Then* $\mathrm{Round}_r(A)$ *is the smallest superset of* $A$ *that is* $r$*-rounded from outside.*

**PROOF.** We already know that $\mathrm{Round}_r(A) \subseteq \mathrm{Round}_r(\mathrm{Round}_r(A))$. Let, conversely, $\mathbf{x} \in \mathrm{Round}_r(\mathrm{Round}_r(A))$, and let furthermore $\mathbf{y} \in \mathbb{R}^n$ be such that $\mathrm{B}(\mathbf{y}, r) \cap A = \emptyset$. Then for all $\mathbf{z} \in \mathrm{Round}_r(A)$ we have $\mathbf{z} \notin \mathrm{B}(\mathbf{y}, r)$, i.e., $\mathrm{B}(\mathbf{y}, r) \cap \mathrm{Round}_r(A) = \emptyset$. Hence $\mathbf{x} \notin \mathrm{B}(\mathbf{y}, r)$, and we have shown that $\mathrm{Round}_r(\mathrm{Round}_r(A)) \subseteq \mathrm{Round}_r(A)$. Let $B \supseteq A$ be $r$-rounded from outside. Then $\mathrm{Round}_r(A) \subseteq \mathrm{Round}_r(B) = B$. □

Compared to roundings from inside, unions and intersections of roundings from outside obey the opposite inclusions. Accordingly, roundings from outside are closed under intersection instead of union.

**Lemma 10** *Let $A$, $B \subseteq \mathbb{R}^n$ be closed, and let $0 < r \in \mathbb{R}$. Then the following assertions hold:*

*(1)* $\text{Round}_r(A) \cup \text{Round}_r(B) \subseteq \text{Round}_r(A \cup B)$
*(2)* $\text{Round}_r(A) \cap \text{Round}_r(B) \supseteq \text{Round}_r(A \cap B)$
*(3) If both $A$ and $B$ are $r$-rounded from outside, then so is $A \cap B$.*

**PROOF.**

(1) Let $\mathbf{x} \in \text{Round}_r(A) \cup \text{Round}_r(B)$, and let $\mathbf{z} \in \mathbb{R}^n$ with $\text{B}(\mathbf{z}, r) \subseteq \sim(A \cup B) = \sim A \cap \sim B$. Then $\mathbf{x} \notin \text{B}(\mathbf{z}, r)$.
(2) Let $\mathbf{x} \in \text{Round}_r(A \cap B)$, and let $\mathbf{z} \in \mathbb{R}^n$ such that $\text{B}(\mathbf{z}, r) \cap A = \emptyset$, respectively $\text{B}(\mathbf{z}, r) \cap B = \emptyset$. Then $\text{B}(\mathbf{z}, r) \cap (A \cap B) = \emptyset$, and thus $\mathbf{x} \notin \text{B}(\mathbf{z}, r)$.
(3) Let $\mathbf{x} \in \text{Round}_r(A \cap B)$, and assume $\mathbf{x} \notin A \cap B$, say $\mathbf{x} \notin A$. Then $\mathbf{x} \notin \text{Round}_r(A)$, and so there exists $\mathbf{z} \in \mathbb{R}^n$ with $\mathbf{x} \in \text{B}(\mathbf{z}, r) \subseteq \sim A \subseteq \sim A \cup \sim B = \sim(A \cap B)$, a contradiction. □

Again, the inclusions are proper in general already in real 2-space: For the first inclusion, consider the two half spaces given by $x_1 \leqslant 0$ and by $x_2 \leqslant 0$. For the second inclusion, consider the complements of the first and of the second quadrant. The first example shows also that roundings from outside are not closed under union.

There appear to be no relations between $r$-roundings from outside and regular closed or open sets. In fact, there is a regular closed, simply connected set $A \subseteq \mathbb{R}^2$ such that $\text{Round}_r(A)$ is disconnected and not regular closed.

The relation between $r$-roundings from inside and $r$-roundings from outside works via the operation of taking closures of complements.

**Theorem 11** *Let $A \subseteq \mathbb{R}^n$ be closed, and let $0 < r \in \mathbb{R}$. Then the following assertions hold:*

*(1)* $\text{round}_r(A) = \overline{\sim \text{Round}_r(\overline{\sim A})}$
*(2)* $\text{Round}_r(A^*)^* = \overline{\sim \text{round}_r(\overline{\sim A})}$.

**PROOF.**

(1) $\text{round}_r(A) = \text{off}_{\leqslant \text{r}}\text{off}_{\geqslant \text{r}}(\overline{\sim A}) = \overline{\text{off}_{<\text{r}}\text{off}_{\geqslant \text{r}}(\overline{\sim A})} = \overline{\sim \text{off}_{\geqslant \text{r}}\text{off}_{\geqslant \text{r}}(\overline{\sim A})} = \overline{\sim \text{Round}_r(\overline{\sim A})}$

(2) We rewrite $\overline{\sim \mathrm{round}_r(\overline{\sim A})}$ according to part (1), and transform it in the following way:

$$\overline{\sim \overline{\overline{\sim \mathrm{Round}_r(\overline{\overline{\sim \overline{\sim A}}})}}} = \overline{\sim \overline{\overline{\sim \mathrm{Round}_r(\overline{\overline{\sim \sim (A^\circ)}})}}} = \overline{\sim \overline{\overline{\sim \mathrm{Round}_r(A^*)}}}.$$

By the same transformation we eventually obtain $\mathrm{Round}_r(A^*)^*$. □

## 7 Comparison to Other Work

Rossignac and Requicha [17] define solids as regular closed sets $A \subseteq \mathbb{R}^n$. Rounding is defined in terms of the offsetting operations of *expanding* $A \uparrow^* r$ and (regularized) *shrinking* $A \downarrow^* r$. The former is exactly our $\mathrm{off}_{\leqslant \mathrm{r}}$. Shrinking is defined in terms of expanding.

$$A \uparrow^* r = \mathrm{off}_{\leqslant \mathrm{r}}(A)$$
$$A \downarrow^* r = \overline{\sim (\overline{\sim A} \uparrow^* r)}.$$

Recall from Corollary 4 that $\mathrm{off}_{\leqslant \mathrm{r}}(A)$ is regular closed for closed $A$. In the definition of shrinking, note that both applications of the closure operator are to open sets. The closures obtained are thus in fact regular closures, i.e., the result of this shrinking applied to closed sets is regular closed. We translate the definition of shrinking to our framework:

$$\begin{aligned} A \downarrow^* r &= \overline{\sim (\overline{\sim A} \uparrow^* r)} = \overline{\sim \mathrm{off}_{\leqslant \mathrm{r}}(\overline{\sim A})} = \overline{\mathrm{off}_{>\mathrm{r}}(\overline{\sim A})} \\ &= \mathrm{off}_{>\mathrm{r}}(\overline{\sim A})^* = \mathrm{off}_{\geqslant \mathrm{r}}(\overline{\sim A})^*. \end{aligned}$$

Rounding from inside and from outside are defined by Rossignac and Requicha as

$$\mathrm{round}'_r(A) = (A \downarrow^* r) \uparrow^* r \quad \text{and} \quad \mathrm{Round}'_r(A) = (A \uparrow^* r) \downarrow^* r,$$

respectively. As we have seen above both expanding and shrinking yields regular closed sets, and so do hence rounding from inside and from outside.

This means that at least $\mathrm{Round}'_r$ must differ in some way from our notion $\mathrm{Round}_r$: Consider, e.g., $\sim ]-r, r[ \subseteq \mathbb{R}$, which is regular closed. In our sense, it is $r$-rounded from outside:

$$\mathrm{Round}_r(\sim ]-r, r[) = \mathrm{off}_{\geqslant \mathrm{r}} \mathrm{off}_{\geqslant \mathrm{r}}(\sim ]-r, r[) = \mathrm{off}_{\geqslant \mathrm{r}}(\{0\}) = \sim ]-r, r[\,.$$

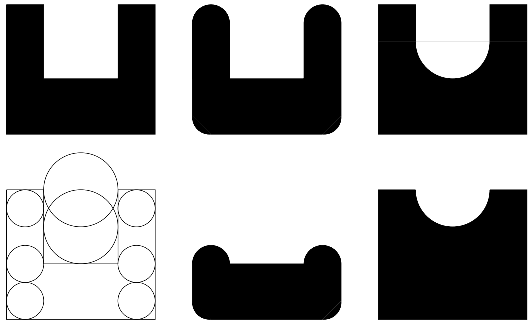

Fig. 1. Given the solid top left, our notions of rounding from inside (center) and outside (right) lead to the results in the top row. Regularized blending yields the results in the bottom row for these limit cases.

In the framework of [17] we obtain in contrast:

$$\mathrm{Round}'_r(\sim]-r,r[) = \mathrm{off}_{\geqslant \mathrm{r}}(\overline{\sim \mathrm{off}_{\leqslant \mathrm{r}}(\sim]-r,r[)})^* = \mathrm{off}_{\geqslant \mathrm{r}}(\emptyset)^* = \mathbb{R}.$$

Similar effects occur with roundings from inside. In our framework, closed balls of radius $r$ are $r$-rounded from inside:

$$\mathrm{round}_r([-r,r]) = \mathrm{off}_{\leqslant \mathrm{r}}\mathrm{off}_{>\mathrm{r}}(\sim[-r,r]) = \mathrm{off}_{\leqslant \mathrm{r}}(\{0\}) = [-r,r].$$

Using $\mathrm{round}'_r$ we obtain in contrast the empty set as inner rounding:

$$\mathrm{round}'_r([-r,r]) = \mathrm{off}_{\leqslant \mathrm{r}}(\mathrm{off}_{\geqslant \mathrm{r}}(\overline{\sim[-r,r]})^*) = \mathrm{off}_{\leqslant \mathrm{r}}(\emptyset^*) = \mathrm{off}_{\leqslant \mathrm{r}}(\emptyset) = \emptyset.$$

The examples can be lifted to any dimension using corresponding balls instead of intervals. Moreover, for a dimension greater than or equal to one, $\sim \mathrm{B}(\mathbf{0},r)$, as used in the example for roundings from outside, becomes connected.

The counterintuitive behavior of regularized roundings has already been observed and illustrated by Rossignac [16] by means of more natural examples similar to Figure 1. Rossignac also defines "non-regularized" variants of rounding from inside and from outside in terms of non-regularized shrinking and expanding:

$$\begin{aligned} A \uparrow r &= \mathrm{off}_{\leqslant \mathrm{r}}(A) \\ A \downarrow r &= \sim(\sim A \uparrow r) = \sim \mathrm{off}_{\leqslant \mathrm{r}}(\sim A) = \mathrm{off}_{>\mathrm{r}}(\sim A) \\ \mathrm{round}''_r(A) &= (A \downarrow r) \uparrow r = \mathrm{off}_{\leqslant \mathrm{r}}\mathrm{off}_{>\mathrm{r}}(\sim A) = \mathrm{round}_r(A) \\ \mathrm{Round}''_r(A) &= (A \uparrow r) \downarrow r = \mathrm{off}_{>\mathrm{r}}(\sim \mathrm{off}_{\leqslant \mathrm{r}}(A)) = \mathrm{off}_{>\mathrm{r}}\mathrm{off}_{>\mathrm{r}}(A) \end{aligned}$$

This rounding from inside amounts to exactly our notion. Non-regularized rounding from outside, in contrast, leads to similar problems as $\mathrm{Round}'$ above.

Rossignac thus defines a “modified” rounding from outside:

$$\begin{aligned}\mathrm{Round}'''_r(A) &= \sim \mathrm{round}''_r((\sim A)^*)^* \\ &= \mathrm{off}_{>\mathrm{r}}\mathrm{off}_{>\mathrm{r}}(\sim(\sim A)^*)^* = \mathrm{off}_{>\mathrm{r}}\mathrm{off}_{>\mathrm{r}}\left(\overset{\circ}{\overline{A}}\right)^* = \mathrm{off}_{\geqslant\mathrm{r}}\mathrm{off}_{\geqslant\mathrm{r}}\left(\overset{\circ}{\overline{A}}\right)^*\end{aligned}$$

For regular closed input $A^*$, we have $\overset{\circ}{\overline{A^*}} = A^{*\circ}$. It follows that Round$'''$ is then the regular closure of our rounding from outside:

$$\mathrm{Round}'''_r(A^*) = \mathrm{off}_{\geqslant\mathrm{r}}\mathrm{off}_{\geqslant\mathrm{r}}(A^{*\circ})^* = \mathrm{off}_{\geqslant\mathrm{r}}\mathrm{off}_{\geqslant\mathrm{r}}(\overline{A^{*\circ}})^* = \mathrm{Round}_r(A^*)^*.$$

Both references discussed finally turn the regularized versions round$'$ and Round$'$ in order to avoid open sets even as intermediate results.

## 8 Computation Examples

We have computed our examples using the package REDLOG [3,5] of the computer algebra system REDUCE [10]. REDLOG has been developed and implemented in Lisp by the first author together with A. Dolzmann. It contains quantifier elimination code for formulas obeying certain degree restrictions. The quantifier elimination method used in REDLOG is based on substitution of test terms. It has been developed by the second author [24]. REDLOG also provides an interface to another quantifier elimination package, QEPCAD by H. Hong, which we use for certain subproblems. QEPCAD uses partial cylindrical algebraic decomposition due to Collins and Hong [2]. Besides quantifier elimination, we make use of sophisticated algebraic simplification methods built into REDLOG [6] when finally substituting concrete radii into the parametric rounding results.

The combination of various programs leads to a certain overhead in the timings. Also, we can only provide wall-clock times. Besides the computation times, we give for each result the number of atomic formulas, i.e. polynomial constraints, contained in it. The polynomials involved are usually not very large, so this gives a good idea of the size of our descriptions. The examples have been computed on a Sun Ultra 1 Model 140 using 32 MB of memory.

REDLOG is a very universal system. It has been successfully applied to various areas in science and engineering [8]:

- Simulation, sizing, and diagnosis of physical networks [18,25].
- Control theory [1].
- Stability analysis for PDE’s [14].

- Automatic theorem proving within the various theories available.
- Geometric reasoning [7].
- Computer aided design, computer vision, and solid modeling [19].
- Collision detection and path finding [19].
- Constraint solving.
- Non-convex parametric linear and quadratic optimization [23], transportation problems [15].
- Parametric scheduling [8].
- Real implicitization of algebraic surfaces [8].
- Computation of comprehensive Gröbner bases [22].
- Implementation of guarded expressions for coping with degenerate cases in the evaluation of algebraic expressions [4].

The procedures are currently by no means tuned to the particular type of input discussed here.

Our example roundings have been computed via iterated offsets according to Theorem 5 and Theorem 8, where the computation of the first offset is usually trivial, i.e. has no rounding effects. For some roundings, we have computed this inner offset by hand, since the automatic results caused violations of the degree restrictions mentioned above in the second step. The corresponding roundings are listed below as mere offset computations.

Also due to violation of the degree restrictions, some other examples failed completely. They are listed at the end of this section. In theory, it is clear how to extend the elimination method to cope with these problems [24].

Note once more that the results obtained below are, in general, parametric $r$-offsets and $r$-roundings, valid for all positive radii.

### *8.1 Roundings from Inside*

#### *8.1.1 First Octant*

We compute the $r$-rounding from inside of the first octant $O_1 = S(x_1 \geqslant 0 \wedge x_2 \geqslant 0 \wedge x_3 \geqslant 0)$. As a first step, we obtain $\mathrm{off}_{>\mathrm{r}}(O_1)$ by applying quantifier elimination to the following description:

$$\forall u_1 \forall u_2 \forall u_3 \Big( \bigwedge_{i=1}^{3} u_i \geqslant 0 \longrightarrow \sum_{i=1}^{3} (x_i - u_i)^2 > r^2 \Big).$$

We automatically obtain the quantifier-free description $x_1 - r \geqslant 0 \wedge x_2 - r \geqslant 0 \wedge x_3 - r \geqslant 0$. From this, we have to compute the $\leqslant$-offset:

$$\exists u_1 \exists u_2 \exists u_3 \Big( \bigwedge_{i=1}^{3} u_i - r \geqslant 0 \wedge \sum_{i=1}^{3} (x_i - u_i)^2 \leqslant r^2 \Big).$$

We automatically obtain the following equivalent quantifier-free description of $\mathrm{round}_r(O_1)$. It contains 19 atomic formulas:

$$\begin{aligned}
& x_1^2 - 2x_1 r + x_2^2 - 2x_2 r + 2r^2 - 2r x_3 + x_3^2 \leqslant 0 \vee \\
& \quad (r - x_3 \leqslant 0 \wedge x_1 - r \geqslant 0 \wedge x_2 - r \geqslant 0) \vee \\
& \quad (r - x_3 \leqslant 0 \wedge x_1^2 - 2x_1 r \leqslant 0 \wedge x_2 - r \geqslant 0) \vee \\
& \quad (r - x_3 \leqslant 0 \wedge x_1 - r \geqslant 0 \wedge x_2^2 - 2x_2 r \leqslant 0) \vee \\
& \quad (r - x_3 \leqslant 0 \wedge x_1^2 - 2x_1 r + x_2^2 - 2x_2 r + r^2 \leqslant 0) \vee \\
& \quad (2r x_3 - x_3^2 \geqslant 0 \wedge x_1 - r \geqslant 0 \wedge x_2 - r \geqslant 0) \vee \\
& \quad (x_1^2 - 2x_1 r + r^2 - 2r x_3 + x_3^2 \leqslant 0 \wedge x_2 - r \geqslant 0) \vee \\
& \quad (x_1 - r \geqslant 0 \wedge x_2^2 - 2x_2 r + r^2 - 2r x_3 + x_3^2 \leqslant 0).
\end{aligned}$$

The complete computation requires less than 1 s.

### *8.1.2 Full Box*

Rounding the full box $B = S(0 \leqslant x_1 \leqslant 2 \wedge 0 \leqslant x_2 \leqslant 3 \wedge 0 \leqslant x_3 \leqslant 4)$ yields within 3 s a quantifier-free description of $\mathrm{round}_r(B)$ containing 106 atomic formulas.

Substituting $r = 2$ and simplifying yields "false" describing the empty set. Substituting $r = 1$ yields a description of the 1-rounded box, which contains only 50 atomic formulas. The time for substitution and simplification can be neglected here (10 ms and 40 ms of CPU time, respectively).

Note that the 1-rounding of this box would be the empty set with the common notion of regularized rounding.

### *8.1.3 Rossignac's Limit Case*

We compute the rounding from inside of Rossignac's solid pictured in Figure 1. it is described by

$$\begin{aligned}
& (0 \leqslant x_1 \leqslant 8 \wedge 0 \leqslant x_2 \leqslant 3) \vee \\
& \quad (0 \leqslant x_1 \leqslant 2 \wedge 3 \leqslant x_2 \leqslant 7) \vee (6 \leqslant x_1 \leqslant 8 \wedge 3 \leqslant x_2 \leqslant 7).
\end{aligned}$$

After 182 s we obtain a quantifier-free description containing 2063 atomic formulas. Substituting $r = 1$ and simplifying requires 16 s. The description then obtained for the 1-rounding contains 1298 atomic formulas.

If we restrict to the 1-rounding from the beginning, we obtain a description with 1265 atomic formulas in 69 s.

### 8.2 Roundings from Outside

#### 8.2.1 Closed Complement of First Octant

For the $r$-rounding from outside of the closed complement of the first octant $S(x_1 \leqslant 0 \vee x_2 \leqslant 0 \vee x_3 \leqslant 0)$ we obtain, similarly to rounding the octant itself from inside, 19 atomic formulas in less than one second.

### 8.3 Offsets

#### 8.3.1 Semi-infinite Cylinder

We compute the $\leqslant$-$r$-offset of $S(x_1^2 + x_2^2 \leqslant 1 \wedge x_3 \geqslant 0)$, i.e. a semi-infinite cylinder of radius 1. We obtain a quantifier-free description with 186 atomic formulas in 9 s.

Alternatively, we consider the $\leqslant$-$r$-offset of $S(x_1^2 + x_2^2 \leqslant (1-r)^2 \wedge x_3 \geqslant r)$, the computation of which is the non-trivial part of $r$-rounding from inside the cylinder above. Here, we obtain 247 atomic formulas within 22 s.

#### 8.3.2 Semi-infinite Cylindrical Hole

For the $\leqslant$-$r$-offset of a semi-infinite cylindrical hole $S(x_1^2 + x_2^2 \geqslant 1 \wedge x_3 \geqslant 0)$, we obtain 248 atomic formulas in 10 s.

For $r$-rounding from inside this hole, we compute the $\leqslant$-$r$-offset of $S(x_1^2+x_2^2 \geqslant (1+r)^2 \wedge x_3 \geqslant r)$. We obtain 193 atomic formulas in 19 s.

#### 8.3.3 Hollow Box

We compute the $\geqslant$-$r$-offset of

$$S((x_1 \leqslant -r \vee x_1 \geqslant 2+r) \wedge (x_2 \leqslant -r \vee x_2 \geqslant 3+r) \wedge$$

$(x_3 \leqslant -r \vee x_3 \geqslant 4 + r)),$

i.e. the $r$-rounding from outside of the hollow box

$$S((x_1 \leqslant 0 \vee x_1 \geqslant 2) \wedge (x_2 \leqslant 0 \vee x_2 \geqslant 3) \wedge (x_3 \leqslant 0 \vee x_3 \geqslant 4)).$$

This takes 1 s. The resulting description contains 80 atomic formulas.

#### *8.3.4 Blending Cylinders with Orthogonal Boxes*

Our aim is to blend the infinite cylinder $S(x_1^2 + x_2^2 \leqslant 1)$ with the semi-infinite box $S(x_3 \leqslant 0 \wedge -1 \leqslant x_2 \leqslant 1)$. For this, we have to compute the $\geqslant$-$r$-offset of the $\geqslant$-r-offset

$$S(x_1^2 + x_2^2 \geqslant (1 + r)^2 \wedge (x_3 \geqslant r \vee x_2 \leqslant -1 - r \vee x_2 \geqslant 1 + r))$$

of their union. We obtain 2476 atomic formulas in 134 s.

Similarly, we can blend this box with a larger cylinder with radius 4. This yields 2840 atomic formulas in 162 s.

### *8.4 Failed Computations*

Automatic blending of two orthogonal circular cylinders currently fails due to a violation of the degree restrictions imposed by our quantifier elimination method. The same happens for blendings of half spaces with oblique cylinders.

## 9 Conclusions

We have proposed to use quantifier-free formulas for representing solids. The operations commonly encountered in solid modelers and, in addition, offsetting and blending can then be performed by quantifier-elimination. This approach naturally copes with open sets as intermediate results allowing more convenient notions of blending in terms of some non-standard offsetting operations. After we have carefully examined the mathematical properties of these offsets, they have turned out as a valuable tool for comparing and analyzing also the notions of blending discussed elsewhere. The mathematical tools required for our method are part of a general-purpose software package by the first author et al. Some promising sample computations based on experimental implementations point on the practical applicability of our approach.

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